A LANDING THEOREM FOR PERIODIC RAYS
OF EXPONENTIAL MAPS

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ABSTRACT. For the family of exponential maps $z \mapsto \exp(z) + \kappa$, we show the following analog of a theorem of Douady and Hubbard concerning polynomials. Suppose that g is a periodic dynamic ray of an exponential map with nonescaping singular value. Then g lands at a repelling or parabolic periodic point. We also show that there are periodic dynamic rays landing at all periodic points of such an exponential map, with the exception of at most one periodic orbit.

1. INTRODUCTION

In polynomial dynamics, *dynamic rays*, which foliate the basin of the superattracting fixed point ∞, play an important role. By a result of Douady and Hubbard, every periodic dynamic ray of a polynomial with connected Julia set lands at a repelling or parabolic periodic point [M1, Theorem 18.10]. The combinatorial information given by this fact was a key element of their initial study of the Mandelbrot set [DH], and has since proved to be one of the most fundamental tools in the field.

In the case of exponential maps $E_\kappa(z) = \exp(z) + \kappa$, the point ∞ is no longer an attracting fixed point but rather an essential singularity, and the set of escaping points is no longer open. However, it has long been known [DGH] that there exist curves of escaping points in the dynamical plane which can be seen as an analog of dynamic rays (compare Section 2). The combinatorics of such rays landing together has also been used to great advantage in this family (see, e.g., [S2, RS1, RS2]).

Therefore it is a natural and important question whether periodic dynamic rays of exponential maps land in general; compare [S1, Section VI.6]. In this article, we give a positive answer to this question.

Theorem 1. *Let $\kappa \in \mathbb{C}$ such that the singular value κ of E_κ does not escape to infinity. Then every periodic dynamic ray of E_κ lands at a repelling or parabolic periodic point.*

REMARK. For maps with escaping singular value, there is an obvious exception if the singular value lies on a periodic dynamic ray (and this is the only exception). Since these parameters were already treated in [SZ2], we have formulated the theorem for nonescaping parameters to simplify its statement.

The analog of Theorem 1 in the polynomial case is proved by a hyperbolic contraction argument. In the exponential family, this proof can be modified to work in the special case

where the ray considered does not intersect the postsingular set [SZ2, R1]. This condition is satisfied for all periodic dynamic rays in the important cases of attracting, parabolic and postsingularly finite parameters; see [SZ2]. For other conditions under which the landing of periodic rays can be proved by an argument in the dynamical plane, compare [R3, R5].

On the other hand, there is a generic subset of the bifurcation locus consisting of maps whose postsingular set is the entire plane [R1, Theorem 5.1.6]. In this case, it is impossible to prove by hyperbolic contraction that a given periodic ray lands, unless one can *a priori* ensure that the ray does not accumulate on the singular value κ. However, the accumulation behavior of dynamic rays can be very complicated in general, e.g., for a large set of parameters there exist rays whose accumulation sets are indecomposable continua [R4]. It even seems difficult to show dynamically, without any structural assumptions, that the accumulation set of a periodic ray cannot be the entire plane.

Our proof of Theorem 1 manages to circumvent these problems by working in the parameter plane, using the "λ-lemma" from the theory of holomorphic motions [MSS] and a theorem of Schleicher concerning the landing of periodic *parameter* rays [S1]. It is somewhat surprising that the landing of dynamic rays is proved using landing properties of parameter rays. Unfortunately, this method will not carry over to higher-dimensional parameter spaces of entire transcendental functions.

We also prove a partial converse to Theorem 1. In the case of a polynomial with connected Julia set, it is known that every repelling or parabolic periodic point is the landing point of a dynamic ray [M1, Theorem 18.11]. The analog of this theorem has previously been shown for attracting, parabolic, escaping and postsingularly finite exponential maps [SZ2]. (In the escaping case, there will again be exceptions when the singular value lies on a periodic ray.)

Theorem 2. *Let $\kappa \in \mathbb{C}$ such that the singular value κ of E_κ does not escape. Then, with the exception of at most one periodic orbit, every periodic point of E_κ is the landing point of a periodic dynamic ray.*

An irrationally indifferent periodic point can never be the landing point of a periodic dynamic ray by the Snail Lemma [M1, Lemma 16.2]. Thus our theorem implies that, in the case where κ is a Siegel or Cremer parameter, all repelling periodic points are landing points. We believe that, as for polynomials, nonrepelling orbits are the only exceptions which can actually occur in Theorem 2. This conjecture can be phrased as a question about parameter space (see Section 5) and would imply that every indifferent parameter κ is the landing point of a parameter ray.

Organization of the article. Naturally, we will rely on a fair number of results now established in the theory of exponential maps. We have attempted to collect these in Sections 2 and 4, so that the proofs of Theorems 1 and 2 will only rely on the propositions stated there.

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Notation and preliminaries. Throughout this article, $\mathbb{C}$ will denote the complex plane, and all closures will be taken in $\mathbb{C}$ (rather than on the Riemann sphere). The set of *escaping points* of an exponential map E_κ is denoted by

$$I(E_\kappa) := \{z \in \mathbb{C} : |E_\kappa^n(z)| \to \infty\}.$$

By [EL], an exponential map has at most one periodic orbit which is not repelling. Therefore, we call a parameter κ *attracting*, resp. *parabolic*, if E_κ has an attracting, resp. parabolic (i.e., rationally indifferent), periodic orbit. As a model of exponential growth, we will fix the function $F : [0, \infty) \to [0, \infty), F(t) = \exp(t) - 1$.

The completion of a proof is marked by the symbol ∎. Results which are cited without proof are concluded by □.

2. Dynamic and Parameter Rays of Exponential Maps

2.1. **Dynamic rays.** We call a sequence $\underline{s} = s_1 s_2 s_3 \dots$ of integers an *external address*. We say that a point $z \in \mathbb{C}$ *has external address* $\underline{s}$ if

$$\text{Im}\big(E_\kappa^{n-1}(z)\big) \in \big((2s_n - 1)\pi, (2s_n + 1)\pi\big)$$

for every $n \geq 1$. The shift map on external addresses is denoted by σ; i.e. $\sigma(s_1 s_2 s_3 \dots) = s_2 s_3 s_4 \dots$. For simplicity, we will restrict to *bounded* external addresses; i.e. those for which the sequence (s_k) of entries is bounded. For the results in the general case, see [SZ1] or [R2].

Proposition 1 ([SZ1]). *Let $\kappa \in \mathbb{C}$, and let $\underline{s}$ be a bounded external address. If $T > 0$ is large enough, then there exists a unique curve $g : [T, \infty) \to I(E_\kappa)$ such that*

(a) *$g(t)$ has external address $\underline{s}$ for all $t \geq T$, and*
(b) *$\text{Re}\big(E_\kappa^n(g(t))\big) = F^n(t) + o(1)$ as n or t tend to ∞.*

This curve has a unique maximal extension $g_{\underline{s}}^\kappa : (t_{\underline{s}}^\kappa, \infty) \to I(E_\kappa)$ satisfying (b). This extension is injective and will be called the dynamic ray *of E_κ at address $\underline{s}$.*

The family of dynamic rays has the following additional properties:

- *$E_\kappa(g_{\underline{s}}^\kappa) \subset g_{\sigma(\underline{s})}^\kappa$ for all $\underline{s}$;*
- *for every $\underline{s}$, $t_{\underline{s}}^\kappa$ depends upper semicontinuously on κ;*
- *for every $t > 0$, $g_{\underline{s}}^\kappa(t)$ depends analytically on κ (where defined);*
- *$t_{\underline{s}}^\kappa \neq 0$ if and only if there exists $n \geq 1$ with $\kappa \in g_{\sigma^n(\underline{s})}^\kappa$ (in which case we call $g_{\underline{s}}^\kappa$ broken).* □

REMARK. In the case of unbounded addresses, the number 0 in the last item is replaced by a number $t_{\underline{s}} \geq 0$ which depends only on $\underline{s}$.

As usual, we say that a dynamic ray $g_{\underline{s}}^\kappa$ *lands* if $t_{\underline{s}}^\kappa = 0$ and $\lim_{t \to 0} g_{\underline{s}}^\kappa(t)$ exists. In our proofs of Theorems 1 and 2, we will use the fact, mentioned in the introduction, that these theorems are known to hold for attracting and parabolic parameters.

Proposition 2 ([SZ2, Theorem 3.2 and Theorem 5.4])**.** *Let κ be an attracting or parabolic parameter. Then every periodic dynamic ray of E_κ lands at a repelling or parabolic periodic point. Conversely, every repelling or parabolic periodic point of E_κ is the landing point of a periodic dynamic ray.* □

REMARK. In fact, for these parameters even more is true: every dynamic ray lands, and every point in the Julia set is either on a ray or the landing point of a ray [BDD, R2].

Parameter rays. By [S1, Theorem II.8.1] (which was recently generalised to the case of unbounded addresses; see [F, FS, FRS]), the set

$$G_{\underline{s}} := \left\{ \kappa \in \mathbb{C} : \kappa \in g_{\underline{s}}^\kappa\big((t_{\underline{s}}^\kappa, \infty)\big) \right\}$$

is itself a continuous (and differentiable) curve to ∞ for every bounded external address $\underline{s}$. This curve is called the *parameter ray* at address $\underline{s}$. We shall require the following result of Schleicher, stating that every periodic parameter ray lands. (The *ray period* of a parabolic parameter κ is the period of the repelling petals of E_κ at the parabolic orbit; in particular, it is a multiple of the period of this orbit.)

Proposition 3 ([S1, Theorem V.7.2])**.** *Suppose that $\underline{s}$ is a periodic address of period n. Then $\overline{G_{\underline{s}}} \setminus G_{\underline{s}}$ consists of a single parabolic parameter of ray period n.*

Conversely, for every parabolic parameter κ of ray period n, there exists a periodic address of period n such that $\kappa \in \overline{G_{\underline{s}}}$.

Schleicher's proof of this proposition consists of two steps. First it is shown that every parabolic parameter is the landing point of one or two parameter rays (whose addresses depend on the combinatorics of the parameter κ). This result is so far unpublished outside of [S1]. It is then shown (independently from the previous step) that, for every periodic address $\underline{s}$, there exists a parabolic parameter with the required combinatorics. A proof of this can be found in [RS1, Theorem 8.5]. □

Vertical order. Finally, let us note that the vertical order of dynamic rays coincides with the lexicographic order $<$ on external addresses. More precisely, let $\underline{s} \neq \underline{r}$ be bounded external addresses, and let $t_0 > t_{\underline{s}}^\kappa$. If $R > 0$ is large enough, then the set

$$\{z : \operatorname{Re} z > R\} \setminus g_{\underline{s}}^\kappa\big([t_0, \infty)\big)$$

has two unbounded components U^+ and U^-, with unbounded positive, respectively negative, imaginary parts. The ray $g_{\underline{r}}$ tends to ∞ in U^- if and only if $\underline{r} < \underline{s}$. (This fact follows easily from the definition of dynamic rays; see [FS, Lemma 3.9] or [R1, Lemma 3.7.1] for details.)

This statement can be transferred to the parameter plane [FS, Proposition 3.10]. That is, if $\underline{s}^1$ and $\underline{s}^2$ are bounded external addresses, then $G_{\underline{s}^1}$ is below $G_{\underline{s}^2}$ in the above sense if and only if $\underline{s}^1 < \underline{s}^2$.

3. Landing of Periodic Dynamic Rays

The key idea in our proof of Theorem 1 is that dynamic rays move holomorphically with respect to the parameter (where defined). By the "λ-lemma" from [MSS], the same is true for their closure. For completeness, we include the short proof of this fact (which is analogous to the argument for the general case in [MSS]).

Theorem 3. *Let $\underline{s}$ be a bounded external address. Let W be a connected component of the set*

$$\mathbb{C} \setminus \overline{\bigcup_{n \geq 1} G_{\sigma^n(\underline{s})}}.$$

Suppose furthermore that for some $\kappa_0 \in W$, $g_{\underline{s}}^{\kappa_0}$ lands. Then $g_{\underline{s}}^{\kappa}$ lands for every $\kappa \in W$, and the landing point depends holomorphically on κ.

REMARK. The theorem (and its proof) remains true for arbitrary external addresses.

PROOF. Let z_0 be the landing point of $g_{\underline{s}}^{\kappa_0}$. By definition of W, the ray $g_{\underline{s}}^{\kappa}$ is unbroken (i.e., $t_{\underline{s}}^{\kappa} = 0$) for every $\kappa \in W$. We shall use the notation $h_t(\kappa) := g_{\underline{s}}^{\kappa}(t)$. The family $(h_t)_{t \in (0,1)}$ of holomorphic functions is normal by Montel's theorem, as it omits the holomorphic functions $\kappa \mapsto \kappa$ and $\kappa \mapsto h_1(\kappa)$. Let h_0 be a limit function of this family as $t \to 0$. Note that $h_0(\kappa_0) = z_0$ and thus, by Hurwitz's theorem, $h_0(\kappa) \neq h_t(\kappa)$ for all $\kappa \in W$ and $t > 0$.

Now if $\widetilde{h_0}$ is some other limit function of h_t as $t \to 0$, then $\widetilde{h_0}(\kappa_0) = z_0 = h_0(\kappa_0)$. Since the h_t omit the holomorphic function h_0, we can apply Hurwitz's theorem once more to see that $\widetilde{h_0} = h_0$. In other words, $h_t \to h_0$ as $t \to 0$. In particular, $g_{\underline{s}}^{\kappa}(t) \to h_0(\kappa)$ for every $\kappa \in W$. ∎

PROOF OF THEOREM 1. Let $\underline{s}$ be any periodic address of period n, and let $\kappa \in \mathbb{C}$ be a parameter with $\kappa \notin I(E_{\kappa})$. Set

$$K := \bigcup_{1 \leq j \leq n} \overline{G_{\sigma^j(\underline{s})}}.$$

By Propositions 3 and 2, we may suppose that $\kappa \notin K$.

Let W be the connected component of $\mathbb{C} \setminus K$ containing κ_0. It is a standard fact that W contains some attracting parameter κ_0. (This follows from Proposition 4 below, but can be proved in a much more elementary way: by [S2, Theorem 3.5], there is a hyperbolic component of parameter space (compare Section 4) tending to ∞ between any two parameter rays.) The ray $g_{\underline{s}}^{\kappa_0}$ lands by Proposition 2, and by Theorem 3, the ray $g_{\underline{s}}^{\kappa}$ also lands. By the Snail Lemma [M1, Lemma 16.2], its landing point is repelling or parabolic. ∎

4. Wakes of Hyperbolic Components

The article [RS2] gives a complete description of the combinatorial structure of *wakes* in exponential parameter space. Proposition 3 and Theorem 3 can be used to translate these results into statements about the regions in which certain periodic rays land together, which will be needed in the proof of Theorem 2. These results, which are all analogs of
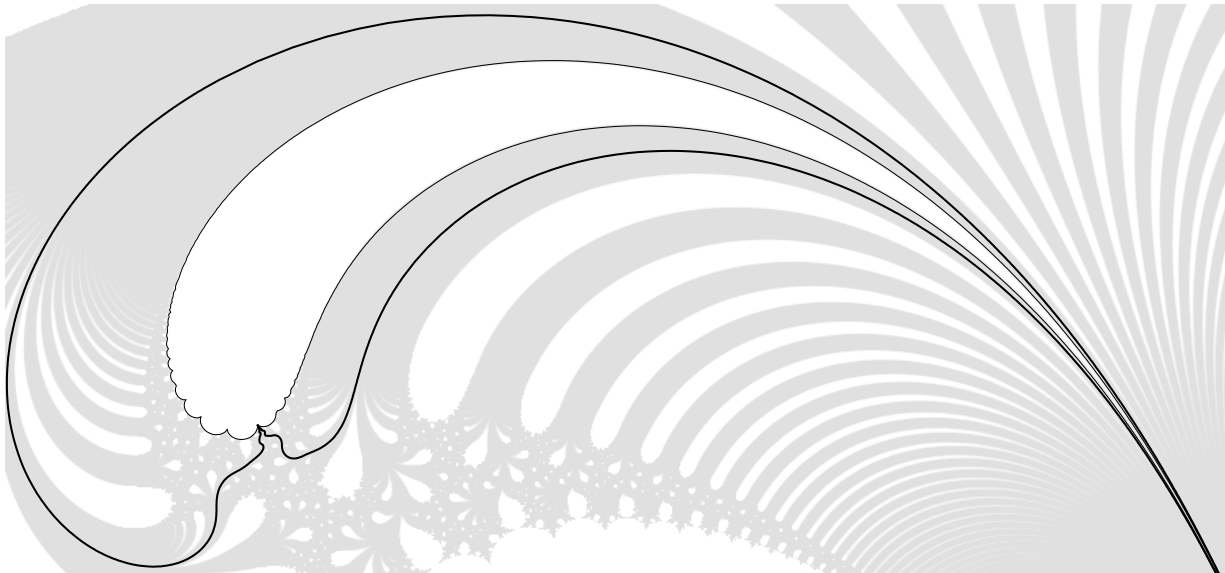
FIGURE 1. The wake of a period 5 hyperbolic component.

well-established results for the Mandelbrot set (see, e.g., [M2]), will be given in this section. Their proofs necessarily involve citing several combinatorial results from [RS2].

A *hyperbolic component* W is a maximal connected subset of parameter space in which all maps have an attracting periodic orbit, the period of which is necessarily constant on W. It is well-known [BR] that there is a unique hyperbolic component of period one, which contains the left half-plane $\{\operatorname{Re}\kappa < -1\}$.

Proposition 4. *Let W be a hyperbolic component of period $n > 1$. Then there exists a unique parabolic parameter $\kappa_0 \in \partial W$ of ray period n (called the* root *of W) which is the common landing point of two periodic parameter rays $G_{\underline{s}^1}$ and $G_{\underline{s}^2}$ of period n. These rays are called the* characteristic rays *of W, and they separate W from the unique hyperbolic component of period one. The connected component of*

$$\mathbb{C} \setminus \big(G_{\underline{s}^1} \cup \{\kappa_0\} \cup G_{\underline{s}^2}\big)$$

which contains W is called the wake *of W and denoted by $\mathcal{W}(W)$. Any two periodic parameter rays which land at a common point are the characteristic rays of some hyperbolic component.* □

REMARK. The unique period one component does not have characteristic rays; by definition, its wake is all of $\mathbb{C}$.

For a proof of Proposition 4, compare Definition 8.6 and the preceding discussion in [RS1]. The wake of a hyperbolic component is important to us because configurations of periodic dynamic rays landing together are preserved throughout it. In order to make this precise, suppose that z_0 is a periodic point of period n for an exponential map E_{κ_0} and U is an open neighborhood of κ_0 in parameter space. By an *analytic extension of z_0 on U* we

mean an analytic function $z : U \to \mathbb{C}$ with $z(\kappa_0) = z_0$ and $E^n_\kappa(z(\kappa)) = z(\kappa)$ for all $\kappa \in U$. By the implicit function theorem and the identity principle, z_0 has at most one analytic continuation on U provided that $(E^n_\kappa)'(z_0) \neq 1$. Furthermore, if U is simply connected and contains no parabolics whose ray period divides n, then z_0 does have an analytic extension to U by the monodromy theorem.

Proposition 5. *Let W be a hyperbolic component and $\kappa_0 \in W$. Suppose that E_{κ_0} has a periodic point z_0 which is the landing point of two different periodic dynamic rays $g^{\kappa_0}_{\underline{s}^1}$ and $g^{\kappa_0}_{\underline{s}^2}$. Then z_0 has an analytic extension z to $\mathcal{W}(W)$, and for all $\kappa \in \mathcal{W}(W)$, $g^\kappa_{\underline{s}^1}$ and $g^\kappa_{\underline{s}^2}$ land at $z(\kappa)$.*

PROOF. By [S2, Lemma 5.2], there exist two periodic addresses $\underline{r}^- < \underline{r}^+$ such that

 (a) $g^\kappa_{\underline{r}^-}$ and $g^\kappa_{\underline{r}^+}$ land at a common point belonging to the orbit of z_0, and
 (b) for all $k \geq 0$ and each $j \in \{1, 2\}$, $\sigma^k(\underline{s}^j) \notin (\underline{r}^-, \underline{r}^+)$.

By [RS2, Proposition 7.4 and Lemma 3.9], $\underline{r}^-$ and $\underline{r}^+$ are the characteristic addresses of some hyperbolic component W' with $W \subset \mathcal{W}(W')$. By (b) and the fact that the vertical order of parameter rays coincides with the lexicographic order on external addresses,

$$\mathcal{W}(W') \cap \left(\bigcup_{k \geq 0, j \in \{1,2\}} \mathcal{G}_{\sigma^k(\underline{s}^j)} \right) = \emptyset.$$

By Theorem 3, for each $j \in \{1, 2\}$ and every $\kappa \in \mathcal{W}(W) \subset \mathcal{W}(W')$, the ray $g^\kappa_{\underline{s}^j}$ lands at a periodic point $z^j(\kappa)$, which depends holomorphically on κ. The maps z^1 and z^2 are both analytic extensions of the periodic point z_0, and must therefore be equal, as required. ∎

Suppose that W is a hyperbolic component and V is a hyperbolic component of larger period than W such that $\partial V \cap \partial W \neq \emptyset$. Then V is called a *child component* of W, and $\mathcal{W}(V)$ is called a *subwake* of W (compare Figure 2).

Proposition 6. *Let W be a hyperbolic component and let V be a child component of W. Then $\partial W \cap \partial V$ consists only of the root κ_1 of V.*

Furthermore, let $\kappa_0 \in W$, let z_0 be a point on the attracting orbit of E_{κ_0} and let U be an open neighborhood of $W \cup \{\kappa_1\}$ such that $U \cap \mathcal{W}(V)$ is connected and z_0 has an analytic extension z to U. Then z extends analytically to $U \cup \mathcal{W}(V)$ and $z(\kappa)$ is the landing point of at least two periodic dynamic rays for every $\kappa \in \mathcal{W}(V)$.

If $W' \neq W$ is a hyperbolic component with $W' \subset \mathcal{W}(W)$, then W' is contained in some subwake of W.

PROOF. According to [RS1, Proposition 8.1], $\partial W \cap \partial V$ contains only parabolic parameters. By [RS2, Proposition 5.1 and Theorem 5.3], no parabolic parameter except κ_1 can belong to this intersection, and there are at least two periodic rays landing at each point of the parabolic orbit of E_{κ_1}. The point $z(\kappa)$ belongs to this parabolic orbit when $\kappa = \kappa_1$, and becomes repelling when κ is perturbed into V.

FIGURE 2. Some subwakes of the period 5 component from Figure 1.

By [RS2, Proposition 5.2], the point $z(\kappa)$ is still the landing point of at least two periodic rays after such a perturbation, and the second statement now follows from Proposition 5. The final claim is [RS2, Corollary 6.9]. ∎

Finally, we shall require the following bound for parameters on parameter rays with large combinatorics [R2, Corollary 4.8]. (Also compare [RS1, Lemma 5.10].)

Proposition 7. *Let $\underline{s}$ be an external address of period n such that $M := \max |s_k| > F^n(6)$. Then*

$$G_{\underline{s}} \subset \left\{ \kappa : |\kappa| > \frac{1}{5} F^{-(n-1)}(2\pi M) \right\}. \qquad \Box$$

5. PROOF OF THEOREM 2

Theorem 4. *Let E_{κ_0} be an exponential map for which the singular value does not escape. Suppose that E_{κ_0} has a periodic point z_0 of period n which is not the landing point of a periodic dynamic ray. Then there exists a hyperbolic component W of period n such that κ lies in $\mathcal{W}(W)$, but not in any subwake of W.*

Furthermore, all period n points of E_{κ_0} which do not lie on the orbit of z_0 are landing points of periodic dynamic rays.

PROOF. By Proposition 2, we can suppose that κ_0 is not a parabolic parameter. Consider the set A consisting of all parameter rays of period dividing n together with the landing points of these rays; by Proposition 3, the latter points are exactly the parabolic parameters whose ray period divides n. We claim that A is closed. Indeed, for every $M \in \mathbb{N}$ the finite

union

$$A_M := \bigcup \left\{ \overline{G_{\underline{s}}} : \underline{s} \text{ is periodic of period dividing } n \text{ and } |s_j| \le M \text{ for all } j \right\}$$

is closed. By Proposition 7, the sequence of sets $(A_M \setminus A_{M-1})$ has no finite accumulation point as $M \to \infty$. Thus $A = \bigcup A_M$ is closed as well.

Now let U be the component of $\mathbb{C} \setminus A$ which contains κ_0. By Proposition 4, U contains a unique hyperbolic component W of period $k|n$, and this component satisfies $U \subset \mathcal{W}(W)$. Since $A \cup \{\infty\}$ is a connected subset of the Riemann sphere, U is simply connected. Thus any period n point z of E_{κ_0} has an analytic extension $z : U \to \mathbb{C}$. Let $\kappa \in W$. We claim that if z is not the landing point of a periodic dynamic ray of E_{κ_0}, then $z(\kappa)$ is on the attracting orbit of E_κ (and in particular, $k = n$). Indeed, suppose that $z(\kappa)$ is not on the attracting orbit of E_κ; then it is a repelling periodic point of E_κ. By Proposition 2, there is a periodic dynamic ray $g_{\underline{s}}^\kappa$ landing at $z(\kappa)$. If more than one periodic ray lands at $z(\kappa)$, then by Proposition 5, $g_{\underline{s}}^{\kappa_0}$ lands at z. Otherwise, $\underline{s}$ has period n. Because U does not intersect any parameter rays of period n, Theorem 3 implies that, again, $g_{\underline{s}}^{\kappa_0}$ lands at z.

So the analytic continuation of z_0 throughout U becomes attracting in W, and every period n point which is not on the orbit of z_0 is the landing point of a periodic dynamic ray of E_{κ_0}. If V is a child component of W, then clearly $\mathcal{W}(V) \cap U$ is connected. Thus $\kappa \notin \mathcal{W}(V)$ by Proposition 6. $\blacksquare$

PROOF OF THEOREM 2. Let z and w be periodic points of E_κ which lie on two different periodic orbits and let n and m be their respective periods. Suppose by contradiction that neither point is the landing point of a periodic dynamic ray of E_κ. By the previous theorem, $n \ne m$ and there exist period n and m hyperbolic components W and V with $\kappa \in \mathcal{W}(W) \cap \mathcal{W}(V)$ so that κ is not contained in any subwake of W or V. Because the two wakes are not disjoint, one is contained in the other; suppose without loss of generality that $\mathcal{W}(W) \subset \mathcal{W}(V)$. By Proposition 6, $\mathcal{W}(W)$ lies in a subwake of V, which is a contradiction. $\blacksquare$

By Theorem 4, the conjecture that every repelling periodic point of an exponential map with nonescaping singular value is the landing point of a periodic dynamic ray is equivalent to the following conjecture about parameter space. (For a discussion, see [R1, Section 7.2].)

Conjecture. *Let W be a hyperbolic component. Then $\mathcal{W}(W)$ is the union of $\overline{W}$, the subwakes of W, and certain parameter rays.*

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